



# Geometric phases in discrete dynamical systems



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## ARTICLE INFO

### Article history:

Received 7 February 2016

Received in revised form 17 August 2016

Accepted 25 August 2016

Available online 31 August 2016

Communicated by A.P. Fordy

### Keywords:

Geometric phase

Circle map

Standard map

Chaos

Superdiffusion

## ABSTRACT

In order to study the behaviour of discrete dynamical systems under adiabatic cyclic variations of their parameters, we consider discrete versions of adiabatically-rotated rotators. Paralleling the studies in continuous systems, we generalize the concept of geometric phase to discrete dynamics and investigate its presence in these rotators. For the rotated sine circle map, we demonstrate an analytical relationship between the geometric phase and the rotation number of the system. For the discrete version of the rotated rotator considered by Berry, the rotated standard map, we further explore this connection as well as the role of the geometric phase at the onset of chaos. Further into the chaotic regime, we show that the geometric phase is also related to the diffusive behaviour of the dynamical variables and the Lyapunov exponent.

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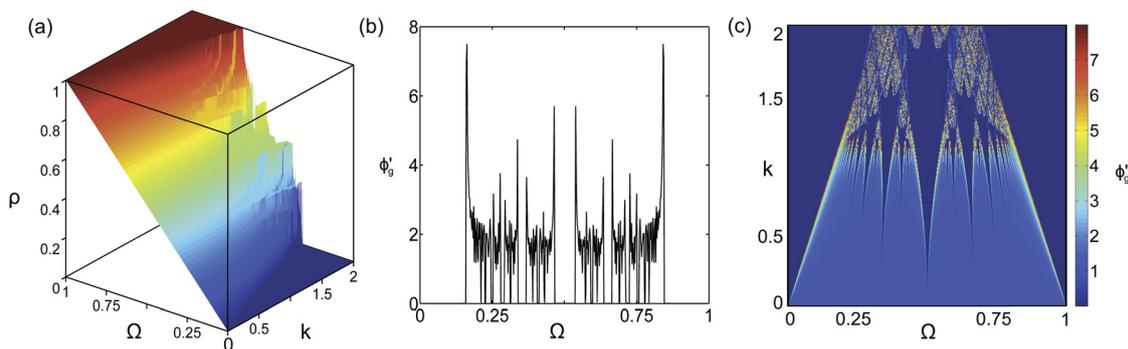
## 1. Introduction

In continuous time dynamics, the study of adiabatic perturbations in general, and of adiabatic cyclic variations in particular, is closely related to the concepts of anholonomy and geometric phase. The geometric phase [1,2] is, indeed, a particular example of anholonomy that we can phrase as the failure of certain variables to return to their original values after a closed circuit in the parameters. Physical expressions of such anholonomies appear in the rotation of the plane of oscillation of a Foucault pendulum [3], how swimming is performed by microorganisms at low Reynolds numbers [4], how the stomach mixes [5], and how a falling cat can manage to reorientate itself in mid air in order to land on its feet [6]. The geometric phase was originally encountered – as Berry's phase – in quantum mechanics [7,8]. From there, it was generalized to classical integrable systems as Hannay's angle [9]. Later it was extended to nonintegrable perturbations of Hamiltonian systems [10–12], and thence to dissipative systems [13–15], all these instances within the context of continuous-time dynamics. In the same context, rotated rotators have been natural models in which

to study this phenomenon because they provide an easy way to control the adiabatic nature of the cyclic variation of the parameter. With this in mind, Berry and Morgan, for instance, investigated the geometric phase of a continuous-time Hamiltonian rotated rotator [16]. In spite of the extensive research that has taken place in the last few decades on geometric phases in a large class of applications, neither the geometric phase nor any of its cognates have been considered hitherto in discrete dynamical systems. Moreover, the general question of how a mapping-defined dynamics behaves under an adiabatic parametric cyclic perturbation has not been addressed until now. Our purpose in this paper is to make good this deficit and to introduce a discrete analogue of the geometric phase and show that it is linked to important aspects of the dynamics of maps. In order to do so, we adopt the paradigm of the rotated rotator but follow an inverse sequence to the historical development. In the first place we deal with the sine circle map that may be thought of as the discretization of a kicked rotator of the type Berry and Morgan studied with the addition of strong dissipation. We consider the results of discrete adiabatically-evolving parameter loops in such a prototypical discrete-time dynamical system. The geometric phase in the rotated circle map, it turns out, is intimately related to the behaviour of the rotation number of the map as a function of the bare frequency parameter. Turning to the Hamiltonian side, we study the rotated standard map, in which we discover surprising relationships between the geometric phase, not

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**Fig. 1.** (a) Devil's quarry plot of rotation number  $\rho$  in the sine circle map; a section through the quarry with  $k$  constant is a devil's staircase. (b) Geometric phase  $\phi_g$  plotted against  $\Omega$  for the rotated critical ( $k=1$ ) sine circle map of Eq. (4). (c) In a colormap, same as in (b) but as a function of both parameters  $\Omega$  and  $k$ .

only with the rotation number as in the former case, but also with the Lyapunov exponent and the diffusive behaviour of both action and phase variables. The reason for our following this inverted developmental sequence is that 1D circle maps are simpler as regards the transition from integrability to chaos, in comparison with the much richer behaviour of the 2D Hamiltonian case.

## 2. The rotated circle map

Before introducing our rotated version of the circle map, let us first recall a few necessary definitions and results on the original non-rotated one. The circle map, usually written as

$$\theta_{n+1} = f_{\Omega,k}^{n+1}(\theta_0) = \theta_n + \Omega - \frac{k}{2\pi} \sin 2\pi \theta_n \pmod{1}, \quad (1)$$

where  $\theta_n = f_{\Omega,k}^n(\theta_0)$  represents the  $n$ th iterate of  $\theta_0$ , which qualitatively describes the dynamics of two interacting nonlinear oscillators, is a one-dimensional discrete mapping that describes how a rotator of natural frequency  $\Omega$  behaves when forced at frequency one through a coupling of strength  $k$ . When  $k=0$  the rotator runs uncoupled at frequency  $\Omega$ , but when  $k>0$  it can lock into a periodic orbit: a resonance with some rational ratio  $p/q$  to the driving frequency. To measure the frequency of the rotator, i.e., the average rotation per iteration of the map, it is useful to define the rotation number

$$\rho = \lim_{N \rightarrow \infty} \frac{\theta_N - \theta_0}{N}. \quad (2)$$

If we plot rotation number  $\rho$  against  $\Omega$  and  $k$  – a few hundred iterations after discarding an initial transient are sufficient to give an accurate value for  $\rho$  – we obtain the devil's quarry [17] illustrated in Fig. 1(a). Periodic orbits with different rational rotation numbers show up as so-called Arnold tongues: flat steps in Fig. 1(a). When  $k < 1$  the map is termed subcritical, and intervals on which the rotation number is constant and rational, where there is a periodic orbit of a particular period, punctuate intervals of increasing rotation number, whereas in supercritical circle maps ( $k > 1$ ) the periodic orbits overlap. Chaos is found in the supercritical circle map as iterates wander between the overlapping resonances. In a critical circle map at  $k=1$ , at every value of  $\Omega$  there is a periodic orbit, and the rotation number increases in a staircase fashion with steps at each rational rotation number and rises in between. The devil's quarry becomes the devil's staircase when we look at a section with  $k$  constant through the quarry. The ordering of periodic orbits in the devil's staircase has been understood in terms of Farey sequences and Stern–Brocot trees [18–21], and the transition to chaos in the circle map is well understood.

Now let us consider the rotated circle map; probably the simplest discrete time system with a discretely and adiabatically varying parameter. To this end we introduce a discrete slowly varying parameter  $X_n$ :

$$\begin{aligned} \theta_{n+1} &= \theta_n + \Omega - \frac{k}{2\pi} \sin 2\pi(\theta_n + X_n) \pmod{1}, \\ X_{n+1} &= X_n + \Delta X = X_n \pm 1/N, \end{aligned} \quad (3)$$

where  $n=1, 2, \dots, N$ , with  $N \rightarrow \infty$  for adiabaticity. Is there a geometric contribution to the phase after such an excursion? Let us perform the change of variable  $\theta'_n = \theta_n + X_n$ , under which the map can be written as

$$\begin{aligned} \theta'_{n+1} &= \theta'_n + (\Omega + \Delta X) - \frac{k}{2\pi} \sin 2\pi \theta'_n \pmod{1} \\ &= \theta'_n + \Omega' - \frac{k}{2\pi} \sin 2\pi \theta'_n \pmod{1}, \end{aligned} \quad (4)$$

where  $\Omega' = \Omega \pm 1/N$ . So it is seen that the effect of the parameter loop is just a shift in the value of  $\Omega$ .

In general, if one takes a system through a parameter loop, one obtains as a result three phases: a dynamic phase, a nonadiabatic phase, and a geometric phase. If one then traverses the same loop in the opposite direction, the dynamic phase accumulates as before, while the geometric phase is reversed in sign. There is, of course, still the nonadiabatic phase too; to get rid of this one must travel slowly around the loop. Thence the geometric phase may be obtained as

$$\phi_g = \lim_{N \rightarrow \infty} \frac{\phi_+ - \phi_-}{2}, \quad (5)$$

where  $\phi_+$  and  $\phi_-$  are, respectively, the total angles  $\theta$  accumulated by travelling around the loop in the positive and negative directions. In terms of the primed variables, we can also define

$$\phi_g' = \lim_{N \rightarrow \infty} \frac{f_{\Omega+1/N,k}^N(\theta_0) - f_{\Omega-1/N,k}^N(\theta_0)}{2}, \quad (6)$$

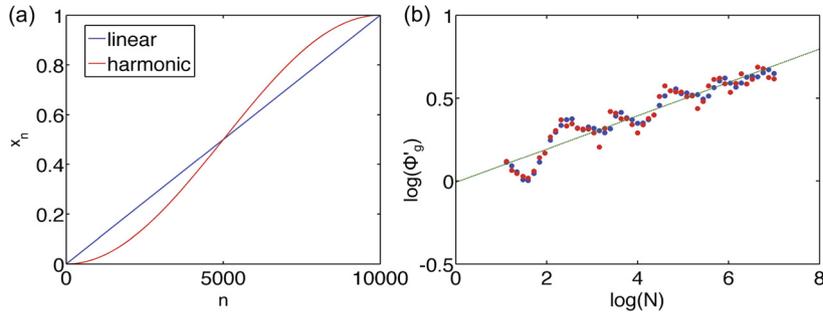
the obvious relation  $\phi_g = \phi_g' - 1$ . Let us evaluate this limit, first for the simple case  $k=0$ . Then  $f_{\Omega,0}^N = \theta_0 + N\Omega$ , so  $f_{\Omega-1/N,0}^N = \theta_0 + N\Omega - 1$  and  $f_{\Omega+1/N,0}^N = \theta_0 + N\Omega + 1$ , hence  $\phi_g' = 1$  and  $\phi_g = 0$ . This limiting case is conceptually equivalent to that of a Foucault pendulum located at the Earth's equator where the plane of oscillations remains fixed as the Earth rotates.

More interesting is what happens when  $k \neq 0$ . From the definition of the rotation number

$$\rho = \lim_{N \rightarrow \infty} \frac{f_{\Omega,k}^N(\theta_0) - \theta_0}{N} = \lim_{N \rightarrow \infty} \rho_{\Omega,k}^N(\theta_0), \quad (7)$$

where we are defining  $\rho_{\Omega,k}^N(\theta_0) = (f_{\Omega,k}^N(\theta_0) - \theta_0)/N$ , we have that

$$\begin{aligned} \phi_g' &= \lim_{N \rightarrow \infty} \frac{f_{\Omega+1/N,k}^N(\theta_0) - f_{\Omega-1/N,k}^N(\theta_0)}{2} \\ &= \lim_{N \rightarrow \infty} \frac{\rho_{\Omega+1/N,k}^N(\theta_0) - \rho_{\Omega-1/N,k}^N(\theta_0)}{2/N}. \end{aligned} \quad (8)$$



**Fig. 2.** (a) The two discretely and adiabatically varying parameter cycles used to illustrate the geometric nature of the phase. (b) In both cases, the geometric phase evaluated through eq. (5) diverges in a similar manner for  $\Omega = \pi/6$  (the line is a linear fit as a guide for the eye).

We may recognize this as a peculiar type of derivative of the rotation number with respect to  $\Omega$  in which the limit  $N \rightarrow \infty$  of the definition of  $\rho$  is taken simultaneously with the limit  $\Delta\Omega \rightarrow 0$ , maintaining  $N\Delta\Omega = 1$ . We may denote such a limit as  $\bar{\partial}\rho/\bar{\partial}\Omega$ . When both limits commute, this derivative coincides with the usual one and

$$\phi'_g = \frac{\bar{\partial}\rho}{\bar{\partial}\Omega} = \frac{\partial\rho}{\partial\Omega}; \tag{9}$$

so that we can say, in this sense, that the geometric phase for the rotated circle map is the derivative of the rotation number. This is obviously true within the tongues where the rotation number is constant and both derivatives zero (Fig. 1(b and c)) and consequently  $\phi'_g = 0$  and  $\phi_g = -1$ . One might think of these cases as equivalents of a Foucault pendulum located at one of the Earth's poles where the plane of oscillations rotates one turn a day in the opposite direction (minus sign) to the Earth's rotation. Outside the tongues both derivatives have singularities arising from the peculiar dependence of  $\rho$  on  $\Omega$ ; nevertheless, apart from convergence (or lack thereof) details,  $\phi'$  clearly has the qualitative aspect of a derivative of the devil's staircase, as illustrated in Fig. 1(b). Fig. 1(c), of which Fig. 1(b) is the  $k = 1$  slice, shows also two more things: for  $k < 1$  where quasi-periodic orbits are still measure-wise abundant the phase outside the tongues is rather regular; this is a consequence of the well known "trivial" scaling relations of the subcritical mode-locking structure [22]. In the supercritical region  $k > 1$  the limit is not well-behaved and fluctuations in the geometric phase appear that correspond to the chaotic regions already noted in Fig. 1(a).

We want to remark on an important difference between the behaviour of our discrete version of geometric phase and its continuous counterpart. While a characteristic feature of the latter is that it can assume any value between zero and  $2\pi$ , in the discrete case it can only take 0 or infinity. One might ask however, whether the places where the divergences occur depend or not on the shape of the adiabatic cycle in the parameter space. Therefore, to assess in this way the geometric nature of the computed phases, we compare in Fig. 2 the results obtained under two different protocols for the discrete slowly varying parameter  $X_n$  over a cycle: the linear constant speed cycle considered above and a nonlinear sinusoidal cycle given by  $X_n = (1 - \cos(n\pi/N))/2$ . Notice that in this case, the connection of our definition of the phase and the derivative of the rotation-number function is no longer analytically obvious. However, while there are clear differences between the corresponding values of the phases evaluated at finite adiabatic velocity variations or cycle lengths  $N$ 's the positions on the  $\Omega$  axis of both the null and diverging phases tend to coincide as the  $N \rightarrow \infty$  limit is approached. A finer comparison of the divergent behaviour as a function of the shape of the cycle should be a subject of further studies.

### 3. Berry's rotated rotator and the standard map

Berry [16] considered the geometric phase of the rotated rotator  $\ddot{q} = V_0 \sin(q - X(t))$ , that is

$$\begin{aligned} \dot{p} &= V_0 \sin(q - X(t)), \\ \dot{q} &= p, \end{aligned} \tag{10}$$

which is a rotator  $\ddot{q} = V_0 \sin q$  being rotated by  $X(t)$ . In the same way that discretizing the rotator with mixed forwards and backwards Euler methods gives us the area-preserving Chirikov–Taylor standard map,

$$\mathbf{T}(I_n, \theta_n) : \begin{cases} I_{n+1} = I_n + (k/2\pi) \sin(2\pi\theta_n) \\ \theta_{n+1} = \theta_n + I_{n+1} \end{cases}, \tag{11}$$

we can consider the rotated standard map that comes from the rotated rotator

$$\begin{aligned} I_{n+1} &= I_n + (k/2\pi) \sin[2\pi(\theta_n + X_n)], \\ \theta_{n+1} &= \theta_n + I_{n+1}, \quad X_{n+1} = X_n \pm 1/N, \end{aligned} \tag{12}$$

where  $X_n$  is again a discretely and adiabatically varying parameter that is moved around a closed loop from 0 to 1.

In the rotating frame of reference, where

$$\theta'_n = \theta_n \pm X_n, \tag{13}$$

the rotated standard map reads:

$$I_{n+1} = I_n + (k/2\pi) \sin[2\pi(\theta_n \pm X_n)], \tag{14}$$

$$\theta'_{n+1} \mp X_{n+1} = \theta'_n \mp X_n + I_{n+1},$$

or

$$\theta'_{n+1} = \theta'_n \pm (X_{n+1} - X_n) + I_{n+1} = \theta'_n \pm \Delta + I_{n+1}. \tag{15}$$

If we define  $I'_n = I_n \pm \Delta$  we obtain the original standard map for the new primed variables:

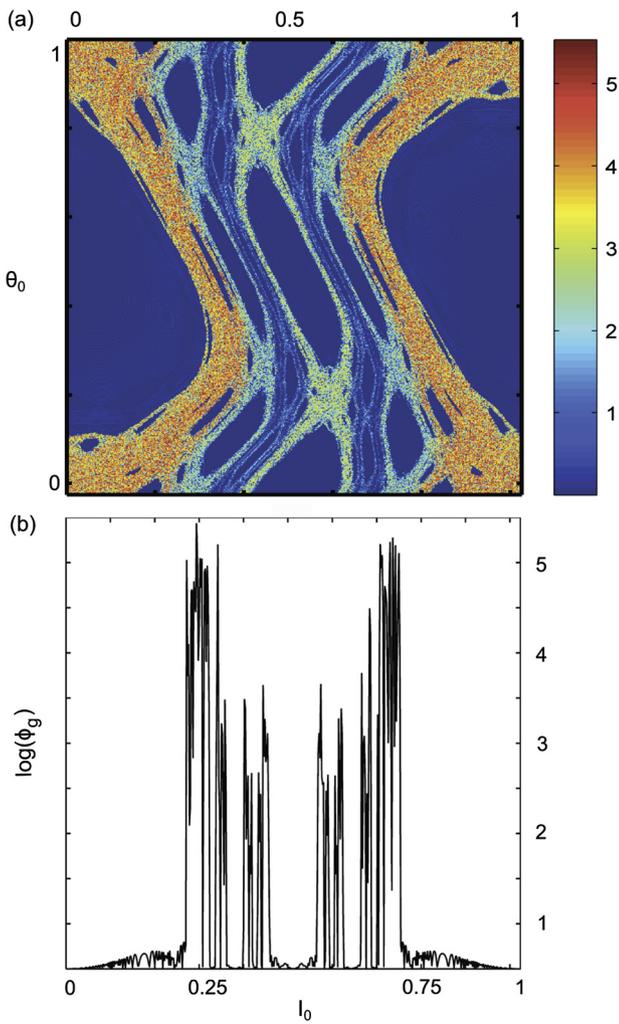
$$\begin{pmatrix} I'_{n+1} \\ \theta'_{n+1} \end{pmatrix} = \begin{pmatrix} I'_n + (k/2\pi) \sin(2\pi\theta'_n) \\ \theta'_n + I'_{n+1} \end{pmatrix}. \tag{16}$$

By the definition of the geometric phase,

$$\phi_g = \lim_{N \rightarrow \infty} \frac{[\mathbf{T}^{+N}]_\theta(I_0, \theta_0) - [\mathbf{T}^{-N}]_\theta(I_0, \theta_0)}{2} \tag{17}$$

and

$$\begin{aligned} \phi'_g &= \tag{18} \\ &\lim_{N \rightarrow \infty} \frac{[\mathbf{T}^N]_\theta(I_0 + \Delta, \theta_0) - [\mathbf{T}^N]_\theta(I_0 - \Delta, \theta_0)}{2} = \\ &\lim_{N \rightarrow \infty} \frac{[\mathbf{T}^N]_\theta(I_0 + 1/N, \theta_0) - [\mathbf{T}^N]_\theta(I_0 - 1/N, \theta_0)}{2}. \end{aligned}$$



**Fig. 3.** The logarithm of the geometric phase for the rotated standard map in a rainbow colour code as a function of the initial conditions  $(I_0, \theta_0)$  for  $k = 1$ . (b) A transversal cut through (a) at  $\theta_0 = 0.5$  demonstrates that the phase is zero in the islands.

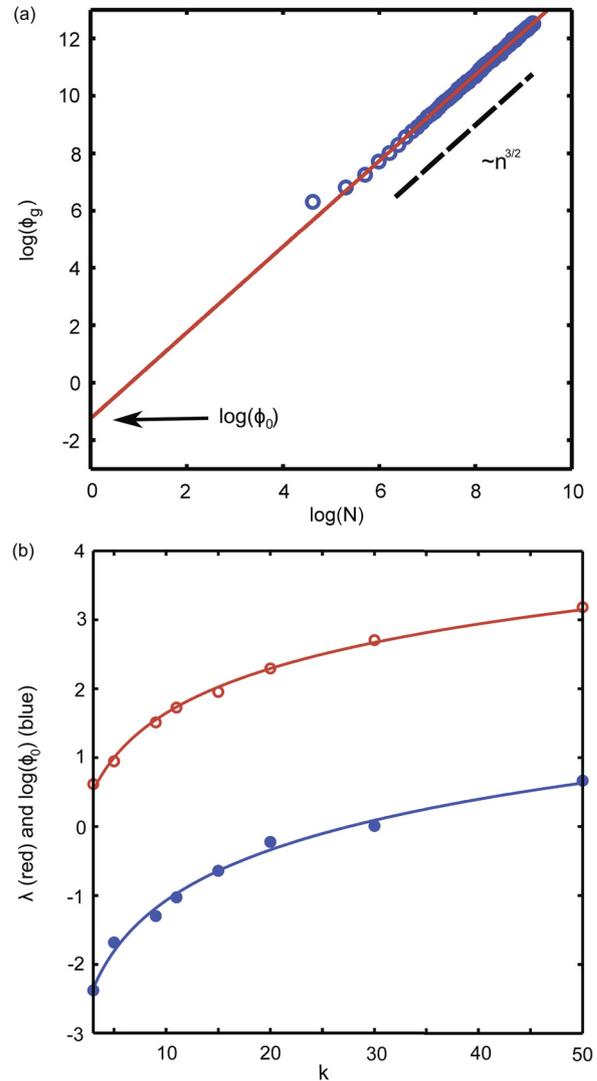
Given that the rotation number  $\rho$  can be expressed as

$$\rho(I_0, \theta_0) = \lim_{N \rightarrow \infty} \rho_N(I_0, \theta_0) = \lim_{N \rightarrow \infty} \frac{[\mathbf{T}^N]_{\theta}(I_0, \theta_0) - \theta_0}{N} \quad (19)$$

we derive the following relation for the geometric phase:

$$\begin{aligned} \phi'_g &= \lim_{N \rightarrow \infty} \frac{\rho_N(I_0 + 1/N, \theta_0) - \rho_N(I_0 - 1/N, \theta_0)}{2(1/N)} \\ &= \frac{\bar{\partial} \rho(I_0, \theta_0)}{\bar{\partial} I_0}. \end{aligned} \quad (20)$$

Of course, this limit does not necessarily exist, nor can one expect it to be finite, so we should take these equations as formal relations. However, as Fig. 3 shows, within the regular KAM islands where the rotation number remains constant, this relation is easily verified and the geometric phase is null. The situation is much less intuitive within the chaotic regions of the phase space; Fig. 3 again. There, the rotation number is not well defined and its dependence on the initial action has not been studied. Naively one could estimate the difference between two neighbouring trajectories in the numerator of the r.h.s. of Eq. (18) as proportional to  $2 \exp(\lambda N)/N$ , which clearly diverges exponentially. On the basis of this assumption one could attempt to establish a connection between this divergence of the geometric phase and the Lyapunov



**Fig. 4.** The geometric phase for the rotated standard map diverges superdiffusively for large  $k$  with power-law exponent  $\beta \approx 3/2$ . (b) The  $y$ -intercept  $\log(\phi_0)$  (filled circles) follows the logarithmic  $k$ -dependence of the Lyapunov exponent,  $\lambda$  (open circles) for large  $k$ .

exponent  $\lambda$  of the trajectory as  $\ln \phi'_g \approx N\lambda + \ln N$ . However, the experimentally computed phase does indeed diverge with  $N$ , but at a much slower pace than exponentially. Instead, we find power-law behaviour for  $\phi_g$  with an exponent of  $3/2$ , which is independent of the value of the nonlinearity parameter  $k$  (Fig. 4(a)). In other words, the phase difference of neighbouring trajectories varying by an amount  $1/N$  in their initial actions grows superdiffusively at the same rate as that of Richardson’s law, after  $N$  iterations. The failure of the exponential assumption is because the Lyapunov exponent only holds if the double limit of the initial perturbation going to zero and the time to infinity is taken in this order. The phase, however, is defined with a similar double limit but taken along the particular direction in which the product of time times the initial perturbation is kept equal to one. On the other hand, the  $3/2$  power law can be explained from the well-known diffusive behaviour of the action due to the chaotic dynamics of the standard map. From the equation for the phase we may realize that the phase difference  $\Delta\theta$  between two trajectories evolves as

$$(\Delta\theta)_{n+1} - (\Delta\theta)_n = (\Delta I)_{n+1}. \quad (21)$$

It follows, for large  $n$ , that

$$\frac{d|\Delta\theta|_n}{dn} \simeq |\Delta I|_n \propto n^{1/2} \Rightarrow |\Delta\theta| \propto n^{3/2}. \quad (22)$$

But there is a further surprising result in the behaviour of the phase illustrated in Fig. 4(b). By extrapolating the power law up to  $\ln N = 0$  we obtain the prefactor of the hypothesized  $3/2$  power law; this prefactor plotted as a function of the nonlinearity amplitude  $k$  behaves very similarly to the Lyapunov exponent plotted as a function of the same parameter, up to an additive constant  $\approx 2$ . The extension, origin and consequences of this connection are under current investigation.

#### 4. Conclusion

In summary, we have shown that the study of adiabatic modulations in maps and, in particular, the extension of the concept of geometric phase to this type of systems opens up a new avenue to understand their dynamical features.

#### Acknowledgements

We acknowledge the financial support of the Spanish Ministerio de Ciencia y Innovación grants FIS2013-48444-C2-1-P & FIS2013-48444-C2-2-P; I.T. acknowledges a Ramón y Cajal fellowship.

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